Low-Momentum-Transfer Pion-Pion Scattering*

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High-energy effects at low momentum transfer, both in the direct and in the crossed channel, are introduced into a self-consistent calculation of low-energy π-π scattering. These effects are assumed to be dominated by top-level Regge trajectories. The general method consists of combining the generalized Chew-Mandelstam and Ball-Wong techniques with self-consistency. It can be extended to complex angular momenta and thereby used to calculate the parameters of the assumed Regge trajectories. A rough self-consistent calculation gives a mass of 712 MeV and a half-width of about 75 MeV for the ρ meson, and a value of 15 mb for the total cross section at very high energies.

I. INTRODUCTION

THERE have been several "bootstrap" methods proposed for calculating the π-π amplitude from the requirements of analyticity, elastic unitarity, and crossing symmetry.[1–9] Of these, only the methods of Chew and Frautschi[5] and of Chew[9] are designed to calculate the high-energy and low-energy amplitudes simultaneously, although numerical calculations have not yet been attempted in either case. In addition, an attempt was made to extend the method of[8] I to make high-energy calculations.[10] However, many high-energy effects were left out in that calculation.

In the present paper, a more general extension of the method of I is given. The original method consisted of setting up an effective-range formula, whose parameters were determined by requiring that its value and derivatives at a suitable matching point be the same as those given by a fixed-energy dispersion relation.[11] The crossed-channel absorptive part coming into this dispersion relation was then approximated by the contributions of a few partial waves, and it was required that the assumed parameters of these waves be equal to the calculated values. Such a calculation is capable of giving a self-sustaining P-wave resonance.

Since only a few partial waves were retained, high-energy effects were completely ignored in the above calculation. It was argued in I that such effects would not make much difference. We shall see that this is a reasonable assumption, at least for calculating the parameters of the ρ meson. On the other hand, Singh and Udgaonkar[12] found that high-energy contributions to the fixed-energy dispersion relation are important in a self-consistent calculation of the N^*. These contributions were taken into account by using the strip approximation, which relates them to low-energy resonances in the direct channel. Thus one is able to make a low-energy calculation which includes high-energy effects in the crossed channel without having to calculate the high-energy amplitude explicitly. This procedure has also been applied to π-Λ scattering by der-Sarkissian[13] and to a self-consistent calculation of the deuteron by Bose and der-Sarkissian.[14]

The main difficulty with the Singh-Udgaonkar approximation is the assumption that the interiors of the double-spectral function regions are unimportant, an approximation which is hard to justify. Moreover, it has been found that one cannot get a reasonable P-wave resonance in the π-π problem if one uses this method, at least if the approximations of I are used. For these reasons, we shall take high-energy effects into account simply by assuming that the high-energy crossed-channel absorptive parts are dominated by top-level Regge poles[15] in the direct channel. This was, in fact, suggested, in I and partially used in Ref. 10. It may be regarded as being in some sense a Regge-pole Singh-Udgaonkar approximation, since it also relates high-energy contributions in the crossed channel to low-energy resonances in the direct channel. Moreover, if one makes certain simplifying assumptions, it leads to a simple result which, although it is quite different from the Singh-Udgaonkar formula for π-π scattering, would reduce to that formula if the scattering particles were sufficiently massive.

Finally, high-energy direct-channel inelastic effects are also included in the present calculation. Here one

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assumes that the elastic amplitude is dominated by Regge poles in the crossed channel. The inclusion of these effects results in a narrowing of the ρ resonance.

II. THE GENERAL APPROACH

We shall begin by reviewing the general approach given in I, using, however, a somewhat different partial-wave amplitude. Specifically, for a given angular momentum l and isotopic spin I, we shall take

$$H_l^I(\nu)=\nu^{-l}(\nu-\nu_K)^{l-1}A_l^I(\nu), \qquad (1)$$

where $\nu=(s/4)-1$, s is the square of the total energy in the barycentric system with pion mass $=1$, and ν_K is a quantity we shall specify later. The function $A_l^I(\nu)$ is the usual π-π partial-wave amplitude[1] for which unitarity gives

$$\mathrm{Im}[A_l^I(\nu)]^{-1}=-[\nu/(\nu+1)]^{1/2}R_l^I(\nu), \qquad (2)$$

where $R_l^I(\nu)=1$ in the elastic region. The amplitude $H_l^I(\nu)$ is chosen so as to give the correct threshold and asymptotic behavior for both physical and unphysical l.[16] Moreover, it does not have the kinematical branch point at $\nu=0$ which is present in $A_l^I(\nu)$ when l is noninteger. It does have a kinematical singularity at $\nu=\nu_K$, but we shall see later that this does not give rise to any difficulties if ν_K is chosen suitably.

If one now makes the usual N/D decomposition,[1]

$$H_l^I(\nu)=N_l^I(\nu)/D_l^I(\nu), \qquad (3)$$

one has, using Eq. (2) and normalizing $D_l^I(\nu)$ to unity at $\nu=\nu_0$,

$$D_l^I(\nu)=1-\frac{\nu-\nu_0}{\pi}\int_0^\infty d\nu'\left(\frac{\nu'}{\nu'-\nu_K}\right)^{l-1}\times\left(\frac{\nu'^3}{\nu'+1}\right)^{1/2}\frac{R_l^I(\nu')N_l^I(\nu')}{(\nu'-\nu_0)(\nu'-\nu)} \qquad (4)$$

and

$$N_l^I(\nu)=\frac{1}{\pi}\int_{-\infty}^{-1}d\nu'\frac{\mathrm{Im}H_l^I(\nu')D_l^I(\nu')}{\nu'-\nu}=\frac{1}{\pi}\int_{\nu_L}^{-1}d\nu'\frac{\mathrm{Im}H_l^I(\nu')D_l^I(\nu')}{\nu'-\nu}-\frac{1}{\pi}\int_0^{x_L}\frac{dx}{x}\frac{\mathrm{Im}H_l^I(-x^{-1})D_l^I(-x^{-1})}{1+x\nu}, \qquad (5)$$

where $x=-\nu^{-1}$, $x_L=-\nu_L^{-1}$, and $\nu_L>-9$. If one now approximates the kernel in the second integral by means of an interpolation formula

$$\frac{1}{1+x\nu}\simeq\sum_{i=1}^{n}\frac{G_i(x)}{1+x_i\nu}, \qquad (6)$$

where the x_i are chosen so as to make the approximation as good as possible, one obtains

$$N_l^I(\nu)=\frac{1}{\pi}\int_{\nu_L}^{-1}d\nu'\frac{\mathrm{Im}H_l^I(\nu')D_l^I(\nu')}{\nu'-\nu}+\sum_{i=1}^{n}\frac{f_i}{x_i^{-1}+\nu}. \qquad (7)$$

The last term has the same form as an n-pole formula.

For given $\mathrm{Im}H_l^I(\nu)$ and $R_l^I(\nu)$, Eqs. (4) and (7) can now be solved to give $H_l^I(\nu)$ through Eq. (3) in terms of the constants f_i. These can then be determined by requiring that this amplitude and $(n-1)$ of its derivatives be the same as that given by

$$A_l^I(\nu)=\frac{1}{\pi\nu}\int_4^\infty dt'\tilde{A}_t^I(t',4(\nu+1))Q_l\left(1+\frac{t'}{2\nu}\right), \qquad (8)$$

at some matching point ν_F in the region $\nu_L<\nu<0$. Equation (8) is obtained by taking a fixed-s dispersion relation for the total amplitude $A^I(s,t)$ and projecting out the lth partial wave. It can also be used for unphysical l.[17] The function $\tilde{A}_t^I$ is given by

$$\tilde{A}_t^I(t,s)=\sum_{I'=0}^{2}\beta_{II'}A_t^{I'}(t,s), \qquad (9)$$

with

$$\beta_{II'}=\begin{bmatrix}\frac{1}{3} & 1 & \frac{5}{3}\\ \frac{1}{3} & \frac{1}{2} & -\frac{5}{6}\\ \frac{1}{3} & -\frac{1}{2} & \frac{1}{6}\end{bmatrix}.$$

Here $A_t^I(t,s)$ is the absorptive part in the t channel, where $t=-2\nu(1-\cos\theta)$ and θ is the scattering angle in the direct channel. Equations (1) and (8) can also be used to obtain $\mathrm{Im}H_l^I(\nu)$, which was needed to solve Eqs. (4) and (7).

If we now take $\nu_K<\nu_L$, we can see why the kinematical singularity at $\nu=\nu_K$ does not cause any difficulty. This singularity is either a pole or a branch point, whose cut can be taken to run to $-\infty$. In either case the net result is to modify $\mathrm{Im}H_l^I(\nu)$ in the region $\nu<\nu_L$, where its explicit form is not needed.

III. THE ABSORPTIVE PARTS IN THE CROSSED CHANNELS

To evaluate Eq. (8), we shall split the integral into two parts,

$$A_l^I(\nu)=A_l^{I(L)}(\nu)+A_l^{I(H)}(\nu), \qquad (10)$$

where

$$A_l^{I(L)}(\nu)=\frac{1}{\pi\nu}\int_4^{t_D}dt'\,\tilde{A}_t^I(t',4(\nu+1))Q_l\left(1+\frac{t'}{2\nu}\right), \qquad (11)$$

$$A_l^{I(H)}(\nu)=\frac{1}{\pi\nu}\int_{t_D}^\infty dt'\,\tilde{A}_t^I(t',4(\nu+1))Q_l\left(1+\frac{t'}{2\nu}\right). \qquad (12)$$

[16] This is not true of $A_l^I(\nu)$ for unphysical l, which is why $H_l^I(\nu)$ was introduced. For large physical l, however, it is probably better to use $A_l^I(\nu)$ and proceed as in I.

[17] M. Froissart (report to the La Jolla Conference on Theoretical Physics, June 1961); V. N. Gribov, Zh. Eksperim. i Teor. Fiz. **41**, 667 and 1962 (1961) [translation: Soviet Phys.—JETP **14**, 478 and 1395 (1962)].

Here t_D is a separation point in the t channel between the low-energy region which is dominated by resonances and the high-energy region which is dominated by Regge poles in the s channel. Now for $0>\nu>\nu_L$, we are within the t-channel Lehmann ellipse, and so the appropriate procedure for $t<t_D$ is to expand in the partial waves of that channel:

$$A_t^I(t,s)=\sum_{l'=0}^{\infty}(2l'+1)\,\mathrm{Im}A_{l'}^I\left(\frac{t}{4}-1\right)P_{l'}\left(1+\frac{2s}{(t-4)}\right). \tag{13}$$

This gives us $A_l^{I(L)}(\nu)$ through Eq. (11).[18]

To find $A_l^{I(H)}(\nu)$ we shall assume that the amplitude for $t>t_D$ is dominated by the top-level Regge pole in the s channel. Lower level poles can be brought in in the same way, however. Thus, in this region we have[15]

$$A^I(s,t)=-\frac{\pi[2\alpha(s)+1]\beta(s)}{2\sin\pi\alpha(s)}\left[P_{\alpha(s)}\left(-1-\frac{2t}{(s-4)}\right)+(-1)^IP_{\alpha(s)}\left(1+\frac{2t}{(s-4)}\right)\right], \tag{14}$$

where $\alpha(s)$ is the position of the pole of $A_l^I(\nu)$ in the l plane and $\beta(s)$ its residue. From Eq. (14) the absorptive part for $t>t_D$ is then

$$\tilde{A}_t^I(t,s)=\tfrac{1}{2}\pi[2\alpha(s)+1]\beta(s)P_{\alpha(s)}(1+2t/(s-4)). \tag{15}$$

To get an explicit expression for $A_l^{I(H)}(\nu)$ we use the fact that, since t_D is large, we can use the approximations

$$P_\alpha(1+2t/(s-4))\simeq C_1(\alpha)(t/2\nu)^\alpha, \tag{16}$$

$$Q_l(1+2t/(s-4))\simeq C_2(l)(t/2\nu)^{-l-1}, \tag{17}$$

where

$$C_1(\alpha)=[2^\alpha\Gamma(\alpha+\tfrac{1}{2})]/[\pi^{1/2}\Gamma(\alpha+1)], \tag{18}$$

$$C_2(l)=[\pi^{1/2}\Gamma(l+1)]/[2^{l+1}\Gamma(l+\tfrac{3}{2})]. \tag{19}$$

If we make these approximations on substituting Eq, (15) into Eq. (12), we obtain

$$A_l^{I(H)}(\nu)=-\nu^l(2\alpha+1)\frac{\beta}{\nu^\alpha}\frac{C_1(\alpha)C_2(l)}{\alpha-l}\left(\frac{t_D}{t_0}\right)^{\alpha-l}, \tag{20}$$

where $t_0=2$. This expression can be continued to regions where the original integral diverges.

The Regge hypothesis also enables us to compute $R_l^I(\nu)$ for $\nu\gtrsim(t_D/4)$ since the amplitude there is dominated by Regge poles in the t and u channels. To obtain their contribution to the partial-wave amplitude, it is convenient to use the representation

$$A_l^I(\nu)=\frac{1}{2}\int_{-1}^{1}dz\,P_l(z)A^I(s,\,-2\nu(1-z))-\frac{\sin\pi l}{\pi}\int_{-\infty}^{-1}dz\,Q_l(-z)A^I(s,\,-2\nu(1-z)), \tag{21}$$

which has been shown by Frazer to be equivalent to Eq. (8).[19] Thus, the contribution of a Regge pole is

$$A_l^I(\nu)=\sum_{I'}\frac{\beta_{II'}}{\nu}\left\{\frac{1}{2}\int_{-4\nu}^{0}dt'P_l\left(1+\frac{t'}{2\nu}\right)A^{I'}(t',4(\nu+1))-\frac{\sin\pi l}{\pi}\int_{-\infty}^{-4\nu}dt'Q_l\left(-1-\frac{t'}{2\nu}\right)A^{I'}(t',4(\nu+1))\right\}, \tag{22}$$

where $A^{I'}(t,s)$ is here understood to be given by Eq. (14). The extra factor of 2 comes from the fact that both t- and u-channel Regge poles are contributing. If we now substitute Eq. (22) into Eq. (2) we can find $R_l^I(\nu)$ for $\nu>(t_D/4)$ in terms of the β and α. The elastic approximation $R_l^I(\nu)\simeq1$ can always be used in the region $\nu<(t_D/4)$.

If we know the β and α, we now see that we have a self-consistency situation for the low-energy partial-wave amplitudes. We can assume certain forms for the $\mathrm{Im}A_l^I(\nu)$, which when substituted into Eq. (13) enable us to evaluate $A_l^{I(L)}(\nu)$ through Eq. (11). This may then be added to Eq. (20) and the general approach of the preceding section may be used to calculate $A_l^I(\nu)$ for $\nu>0$. Self-consistency then requires that the assumed forms for $\mathrm{Im}A_l^I(\nu)$ be the same as these calculated forms. This should be sufficient to determine the low-energy amplitude.

IV. EVALUATION OF REGGE-POLE TRAJECTORIES

In the preceding section a method for obtaining the low-energy amplitude self-consistently was given, assuming the values of $\beta(s)$ and $\alpha(s)$ for small negative values of ν. However, these functions can also be calculated if we follow the procedure of the preceding two sections for unphysical values of l. Then a point $l=\alpha(s_p)$ on the Regge trajectory can be found by calculating the value $s=s_p$ for which

$$D_{\alpha(s_p)}^I(\nu_p)=0, \tag{23}$$

where $\nu_p=(s_p/4)-1$. The value of $\beta(s_p)$ can be deduced by first finding the residue of the corresponding pole of $H_l^I(\nu)$ in the ν plane for $l=\alpha(s_p)$. This is just

$$\Gamma_{\alpha(s_p)}^I=-N_{\alpha(s_p)}^I(\nu_p)[\partial D_{\alpha(s_p)}^I(\nu)/\partial\nu]_{\nu=\nu_p}^{-1}. \tag{24}$$

The residue of the corresponding pole in $A_l^I(\nu)$ in the

[18] An alternative procedure would be to use some low-energy Regge expansion, such as the one proposed by N. Khuri, Phys. Rev. **130**, 429 (1963). This would probably be much more complicated to use, however.

[19] W. Frazer (unpublished—the result is presented in Ref. 9).

l plane is then

$$\beta(s_p)=\nu_p{}^{\alpha(s_p)}(\nu_p-\nu_K)^{1-\alpha(s_p)}\Gamma_{\alpha(s_p)}{}^I[d\alpha(\nu)/d\nu]_{\nu=\nu_p}. \quad (25)$$

The above method can, of course, only be used for $\nu>\nu_L$, since there is no way of finding $N(\nu)$ for $\nu<\nu_L$. Moreover, in practice we can calculate only a few points on the trajectory anyway. However, we can always extrapolate $\alpha(s)$ and $\beta(s)$ away from these points. One way of doing this is to use the fact that for nonintersecting trajectories we can always write[20]

$$\alpha(s)=a_0+\frac{1}{\pi}\int_4^\infty ds'\frac{\mathrm{Im}\alpha(s')}{s'-s}, \quad (26)$$

$$b(s)=b_0+\frac{1}{\pi}\int_4^\infty ds'\frac{\mathrm{Im}b(s')}{s'-s}, \quad (27)$$

where $b(s)=\nu^{-\alpha(s)}\beta(s)$, and a_0 and b_0 are real constants. Since we are only interested in values of $s<4$, we may make the substitution $x=s'^{-1}$, and approximate the kernels in Eqs. (26) and (27) in the same way that we approximated the kernel in the N function. This leads to

$$\alpha(s)=a_0+\sum_{i=1}^{m}\frac{a_i}{\nu_i-\nu}, \quad (28)$$

$$b(s)=b_0+\sum_{i=1}^{m}\frac{b_i}{\nu_i-\nu}, \quad (29)$$

where the ν_i are determined by making the kernel approximation as good as possible. Moreover, as discussed by Chew,[9] the fact that $\mathrm{Im}\alpha(s)$ and $\mathrm{Im}b(s)$ are small over a fairly large range above threshold effectively shifts the lower limit on the integral to a much higher value. Thus, it is necessary to approximate the kernel in a much smaller region, which reduces the number m considerably. The real constants a_i and b_i can then be calculated by fitting Eqs. (28) and (29) to m points on the trajectory. These points would be determined in the manner described in the preceding paragraph. The most convenient values are those which lie in the region $\nu_L<\nu<0$, since $\alpha(s)$ and $b(s)$ are real in this region. In practice, however, we may choose any values for which $\mathrm{Im}\alpha$ and $\mathrm{Im}b$ are small.

If we require that the calculated parameters of α and β be the same as the assumed values, we should be able to calculate these functions. Now in the last section we saw how the low-energy amplitude can be calculated at the same time. Thus, we have an extended "bootstrap" situation, whereby we can calculate self-consistently both the low-energy partial-wave amplitudes and the parameters of the Regge trajectories simultaneously. The latter, of course, give the low-momentum transfer amplitude at high energies.[15]

[20] V. Singh, Phys. Rev. **127**, 632 (1962).

V. p-WAVE RESONANCE

We shall now use a rather crude version of the above method to make a self-consistent calculation of the ρ meson. In evaluating $A_1{}^{1(L)}(\nu)$, we drop everything in the expansion (13) except a zero-width p-wave resonance at $\nu=\nu_R$. This means that the left-hand cut starts at $\nu=-\nu_R-1$. Thus, if we take $\nu_L=-\nu_R-1$, the integral in Eq. (7) will be zero.

The kernel approximation (6) will be made exactly as in I, i.e., we use a straight-line interpolation through $x_1=0.16$ and $x_2=0.02$, for which $n=2$. The zero-width resonance is also set up as in I, i.e., we put $\mathrm{Re}D_1{}^1(\nu)\simeq(\nu-\nu_R)/(\nu_0-\nu_R)$ and $N_1{}^1(\nu)\simeq N_1{}^1(\nu_R)$, which leads to

$$\mathrm{Im}H_1{}^1(\nu)=\frac{(\Gamma_1{}^1)^2[\nu^3/(\nu+1)]^{1/2}}{(\nu-\nu_R)^2+(\Gamma_1{}^1)^2[\nu^3/(\nu+1)]}, \quad (30)$$

with

$$\mathrm{Re}D_1{}^1(\nu_R)=0 \quad (31)$$

and

$$\Gamma_1{}^1=(\nu_R-\nu_0)N_1{}^1(\nu_R). \quad (32)$$

Equation (30) can now be approximated by a delta function, the integral over which is equal to the integral over Eq. (30) in the limit of small $\Gamma_1{}^1$. This gives

$$\mathrm{Im}H_1{}^1(\nu)=\pi\Gamma_1{}^1\delta(\nu-\nu_R), \quad (33)$$

which, together with Eqs. (9), (11), and (13), leads to

$$A_1{}^{1(L)}(\nu)=12\beta_{11}\frac{\nu_R\Gamma_1{}^1}{\nu}\left(1+2\frac{\nu+1}{\nu_R}\right)Q_1\left(1+2\frac{\nu_R+1}{\nu}\right). \quad (34)$$

To evaluate $A_1{}^{1(H)}(\nu)$, instead of using Eqs. (28) and (29) we shall make the usual cruder but simpler assumptions that[21]

$$C_1(\alpha)(2\alpha+1)(\beta/\nu^\alpha)\approx\text{const}, \quad (35)$$

$$\mathrm{Re}\alpha\approx1+\epsilon(\nu-\nu_A), \quad (36)$$

where ϵ and ν_A are constants (see Fig. 1). For the ρ trajectory, which dominates in the $I=1$ state which we are considering here, $\nu_A=\nu_R$. Then if we equate the left-hand side of Eq. (35) to its value at $\nu=\nu_A$, neglect $\mathrm{Im}\alpha$, and use Eq. (25) to evaluate β, we get

$$A_1{}^{1(H)}(\nu)=\frac{\nu\Gamma_1{}^1}{\nu_R-\nu}\left(\frac{t_D}{t_0}\right)^{-\epsilon(\nu_R-\nu)}. \quad (37)$$

It is interesting to note that the first factor in this expression is exactly the same as the one that would be obtained by using the Singh-Udgaonkar approximation[12] and keeping only the contribution of a zero-width p-wave resonance to the s-channel absorptive part. Since $t_0\ll t_D$, however, the second factor is not even roughly unity. This may explain why the Singh-

[21] B. M. Udgaonkar and M. Gell-Mann, Phys. Rev. Letters **8**, 346 (1962). See also Ref. 15.

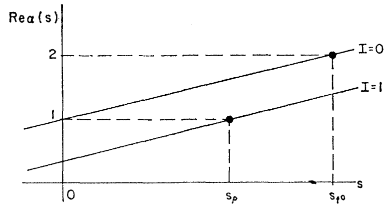

FIG. 1. Schematic plots of the Pomeranchuk ($I=0$) and the $\rho(I=1)$ trajectories, assuming that Eq. (36) holds. The positions s_ρ and s_{f^0} represent the positions of the ρ and f^0 resonances, respectively.

Udgaonkar approximation was not applicable to the π-π problem. For the π-N and N-N problems, however, t_0 is much larger, and so that approximation should be quite reasonable.

Inelastic effects for $\nu \gtrsim (t_D/4)$ can be estimated by the method given in the third paragraph of Sec. III. This is done in the Appendix, where the approximations made are described in detail. One finally obtains

$$R_1{}^1(\nu) \simeq \frac{\pi^2}{\sigma_t} \frac{\nu(\epsilon \ln 2\nu)^2}{\nu\epsilon \ln 2\nu - 2}, \qquad (38)$$

where σ_t is the total cross section at high energies. Equation (38) has the form shown in Fig. 2. To simplify the calculation further this was then approximated by

$$R_1{}^1(\nu) \simeq 1 + \left(\frac{\nu+1}{\nu}\right)^{1/2} \left[R_1{}^1\left(\frac{t_D}{4}\right) - 1\right] \theta\left(\nu - \frac{t_D}{4}\right), \qquad (39)$$

where θ is the usual step function. The factor $[(\nu+1)/\nu]^{1/2}$ is essentially unity and is inserted only to simplify the evaluation of a certain integral. If we now substitute Eqs. (7) and (39) into Eq. (4), and remember that the integral in Eq. (7) is zero, we have

$$D_1{}^1(\nu) = 1 - (\nu - \nu_0) \sum_{i=o}^{n} \left\{ I_{\rm el}(\nu, x_i^{-1}) + \left[R_1{}^1\left(\frac{t_D}{4}\right) - 1\right] I_{\rm in}(\nu, x_i^{-1}) \right\} f_i', \qquad (40)$$

where

$$I_{\rm el}(\nu,\omega) = \frac{2}{\pi(\nu+\omega)} \left\{ \left(\frac{\omega}{\omega-1}\right)^{1/2} \ln[\omega^{1/2} + (\omega-1)^{1/2}] - \left(\frac{\nu}{\nu+1}\right)^{1/2} \ln[\nu^{1/2} + (\nu+1)^{1/2}] \right\}, \qquad (41)$$

$$I_{\rm in}(\nu,\omega) = \frac{1}{\pi(\nu+\omega)} \left[\ln\left(\frac{t_D}{4} + \omega\right) - \ln\left(\frac{t_D}{4} - \nu\right) \right], \qquad (42)$$

and $f_i' = (1 + x_i\nu_0)^{-1} f_i$ for $i > 0$ while

$$f_0' = \sum_{i=1}^{n} (f_i - f_i'),$$

with $x_0 = -\nu_0^{-1}$.

In what follows, we shall take $\nu_0 = \nu_F = -2$ as in I. We shall assume that the $I=0$ and $I=1$ Regge trajectories have the same value of ϵ. If we also assume that for the $I=0$ trajectory,[22] $\nu_A = -1$ and that the recently discovered f^0 particle[23] lies on it with spin 2, we obtain $\epsilon = 1/20$. From the factorization theorem it has been deduced[24] that $\sigma_t = 15$ mb, which gives $R_1{}^1(t_D/4)$ through Eq. (38). In choosing t_D, we assume that the Regge behavior sets in immediately above the resonance region. Since the f^0 is the highest known resonance in the π-π system, we shall thus take $t_D = 80$. If we now assume certain values for $\Gamma_1{}^1$ and ν_R we can calculate the f_i in the manner described above. This, in turn, can be used to find $\Gamma_1{}^1$ and ν_R through Eqs. (7), (40), (31), and (32). We can then vary the assumed $\Gamma_1{}^1$ and ν_R until these are equal to the calculated values. Since none of the above equations entail any numerical integration, this can be readily done by hand and leads to $\nu_R \Gamma_1{}^1 = 1.6$ and $\nu_R = 5.5$, which corresponds to a mass of 712 MeV. A plot of the partial-wave cross section,

$$\sigma_1{}^1 = 12\pi[\nu/(\nu+1)]^{1/2} \, {\rm Im} H_1{}^1(\nu), \qquad (43)$$

has a half-width of 75 MeV if we use Eq. (30). These values should be compared with those deduced from pion-production experiments,[25] which range from 725 to 770 MeV for the mass, and 30 to 75 MeV for the half-width.

It is interesting to note that if we let $t_D \to \infty$, the above problem reduces exactly to the one in I, where high-energy effects were completely ignored. The mass

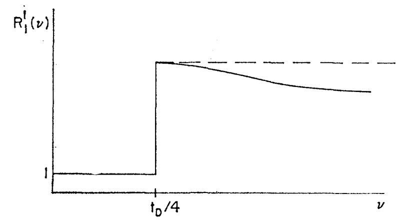

FIG. 2. The function $R_1{}^1(\nu)$. The elastic approximation is made for $\nu < (t_D/4)$. For $\nu > (t_D/4)$ the solid line is a schematic plot of Eq. (38). The function first drops somewhat but eventually rises to infinity logarithmically. The dashed line is a plot of Eq. (39). The very high-energy contribution is unimportant.

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[22] This is required by the Chew-Frautschi saturation principle. See Refs. 5 and 15.

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and half-width obtained there were 585 and 110 MeV, respectively, if we use Eqs. (30) and (43). These numbers do not differ too much from the ones obtained above and so the calculation is certainly insensitive to the choice of t_D. This also justifies the assumption made in I that high-energy effects can be neglected as a first approximation, at least in the π-π problem. But, as already pointed out, such effects can be quite important in other problems.

VI. HIGH-ENERGY CROSS SECTION

In the above calculation σ_t was taken from experiment. However, it can always be calculated by going to the $I=0$, $l=1$ unphysical "state."[10] This time the Pomeranchuk ($I=0$) trajectory dominates in $A_1^{0(H)}(\nu)$, and so, in Eq. (36), we take $\nu_A=-1$.[22] Then, if we use the relation $\sigma_t=4\pi^2\beta(0)$, which can be deduced from Eq. (14) with the help of the optical theorem, but otherwise follow the same procedure as for the ρ trajectory, we get

$$A_1^{0(H)}(\nu)=\frac{-\nu\sigma_t}{4\pi^2\epsilon(1+\nu)}\left(\frac{t_D}{t_0}\right)^{\epsilon(1+\nu)}. \tag{44}$$

Assuming that the ρ is the dominant low-energy contribution in the crossed channel, $A_1^{0(L)}(\nu)$ will have the same form as Eq. (34), except that β_{11} has to be replaced by β_{01}. For ν_R and Γ_1^1 we shall take the values calculated in the previous section. The function $R_1^0(\nu)$ can be easily seen to be exactly the same as $R_1^1(\nu)$.

If one now calculates the f_i as before, one finds that, to within about 5%, $D_1^0(-1)=0$. But from Eq. (23) this just means that the calculated $\alpha(s)$ is unity at $s=0$. Thus, $\nu_A=-1$ is the self-consistent value for $I=0$ and so the Chew-Frautschi saturation condition[22] comes out of the calculation. If one then calculates $\sigma_t=4\pi^2\beta(0)$ through Eqs. (25) and (32) one obtains $\sigma_t=15$ mb almost exactly. This means that the self-consistent value for σ_t is essentially equal to the "experimental" value deduced from the factorization theorem.[24]

The only remaining undetermined parameters in the problem are ϵ and t_D. Now ϵ could be calculated self-consistently by going to more unphysical "states," but this would complicate the calculation considerably. However, the value chosen here was such that it gave the saturation condition correctly. Thus, if we assume that condition, ϵ may be said to have been determined also. This leaves t_D, which, however, is not an arbitrary parameter in the usual sense, since it merely separates two regions within which two different approximations are made. In fact, if the low-energy region is systematically improved by bringing in more partial waves, increasing n, and inserting intermediate-energy inelastic effects, one should increase t_D at the same time. In other words, t_D is merely the point at which the low-energy approximations can be expected to break down. This can generally be estimated on an *a priori* basis. For instance, if only low-energy resonances are kept, as in the above, the strip-width estimate of Chew[9] can always be used, since t_D corresponds to the width of the strip in Ref. 9. It gives $t_D\approx 4\epsilon^{-1}$. If one assumes this estimate, one may say that t_D has also been determined, since the value chosen for t_D did satisfy this rough relation.

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APPENDIX: INELASTIC EFFECTS IN THE p WAVE

At very large values of s, the dominant contribution to the amplitude will come from the Pomeranchuk Regge pole in the t and u channels.[15] From Eqs. (14) and (16), the t-channel contribution for $s>t_D$ is

$$A^I(t,s)=\beta(t)\frac{\pi[2\alpha(t)+1)]}{2\sin\pi\alpha(t)}\times[1+e^{-i\pi\alpha(t)}]C_1[\alpha(t)]\left(\frac{s}{2\nu_t}\right)^{\alpha(t)}, \tag{A1}$$

where $\nu_t=(t/4)-1$. If we make the assumptions (35) and (36), but this time in the t channel, and equate the left-hand side of Eq. (35) to its value at $\nu_t=\nu_A=-1$, we get

$$A^I(t,s)=3\beta(0)(s/2)^{1+\frac{1}{4}\epsilon t}\{\tan\tfrac{1}{8}\pi\epsilon t-i\} \tag{A2}$$

Now, because of the factor

$$(\tfrac{1}{2}s)^{\frac{1}{4}\epsilon t}=\exp[\tfrac{1}{4}\epsilon t\ln(\tfrac{1}{2}s)],$$

$A^I(t,s)$ will be important only in the region

$$|t|(\tfrac{1}{4}\epsilon)\ln(\tfrac{1}{2}s)\lesssim 1,$$

which for $s>t_D$ corresponds to $|t|\lesssim 20$. But in this region the approximation $\tan(\frac{1}{8}\pi\epsilon t)\simeq(\frac{1}{8}\pi\epsilon t)$ is valid to about 10%. Making, therefore, this approximation and setting $(s/2)\simeq 2\nu$, we obtain

$$A^I(t,s)=6\nu\beta(0)\exp[\tfrac{1}{4}\epsilon t\ln 2\nu]\{\tfrac{1}{8}\pi\epsilon t-i\}. \tag{A3}$$

If this is inserted into Eq. (21), the resulting integral can be carried out exactly and gives

$$A_1^1(\nu)=\beta(0)[a_R+ia_I], \tag{A4}$$

where

$$a_R=\tfrac{1}{8}\pi\epsilon(I_1+I_2/2\nu), \tag{A5}$$

$$a_I=I_0+I_1/2\nu, \tag{A6}$$

$$I_0=4\frac{1-\exp[-\nu\epsilon\ln 2\nu]}{\epsilon\ln 2\nu}, \tag{A7}$$

$$I_1=-4\frac{I_0-4\nu\exp[-\nu\epsilon\ln 2\nu]}{\epsilon\ln 2\nu}, \tag{A8}$$

$$I_2=-8\frac{I_1+8\nu^2\exp[-\nu\epsilon\ln 2\nu]}{\epsilon\ln 2\nu}. \tag{A9}$$

If we note that, for $s>t_D$, $[\nu/(\nu+1)]^{1/2}\simeq 1$, we obtain from Eq. (2),

$$R_1^1(\nu)=\frac{4\pi^2}{\sigma_t}\frac{a_I}{a_R^2+a_I^2}, \tag{A10}$$

where we have used the relation $\sigma_t=4\pi^2\beta(0)$, which can be obtained by combining Eq. (14) with the optical theorem. To simplify our result further we could drop all the exponential terms in Eqs. (A7), (A8), and (A9). This does not affect the final result very much and corresponds to setting the lower limit -4ν equal to $-\infty$ in Eq. (21). Such a procedure is not unreasonable, since $4\nu\gtrsim 80$ with $s>t_D$, and $A^I(t,s)$ is important only in the region $|t|\lesssim 20$, as we have seen. A further simplification results from the *a posteriori* observation that $a_R^2\ll a_I^2$. If we make all these approximations, we finally obtain

$$R_1^1(\nu)\simeq\frac{\pi^2}{\sigma_t}\frac{\nu(\epsilon\ln 2\nu)^2}{\nu\epsilon\ln 2\nu-2}. \tag{A11}$$

With $\sigma_t=75$ mb this gives $R_1^1(t_D/4)=5.31$. If we did not drop a_I we would get $R_1^1(t_D/4)=5.28$. These two are practically indistinguishable. The latter value was the one actually used in the calculations of Secs. V and VI.

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Three-Meson Model for *p-p* Scattering and Regge Poles

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By considering a dispersion relation for that amplitude of *p-p* scattering, whose imaginary part in the forward direction is related to the total cross section, it is shown that the one-meson-exchange model (taking into account independent exchanges of the pion, the ρ-ω vector pair, and an $l=0$, scalar 2π resonance or meson with a mass somewhat greater than two pion masses) and high-energy behavior of the *p-p* and *p-p̄* scattering cross sections as given by the Regge pole hypothesis, are consistent with the existing *p-p* scattering data. In our demonstration the energy range involved is larger than previously used in the demonstration of either of the above two aspects of *p-p* scattering. Further by considering a dispersion relation and high-energy behavior of another amplitude of *p-p* scattering, it is shown that the second type of coupling of the Pomeranchuk pole is zero. This reduces the number of unknown parameters in the expression for polarization at high energy.

1. INTRODUCTION

It has been shown by several authors[1–4] that the so called one-meson-exchange model, taking into account independent exchanges of one pion, one η, one ρ, and one ω only, gives an excellent approximation to the experimentally observed nucleon-nucleon scattering. In Refs. 2, 3, and 4 it has been necessary to postulate the existence of an $l=0$ scalar meson or resonance of mass 3 to 4 m_π with a rather large coupling constant with nucleon. So far, there appears to be contradictory evidence on the existence of such a meson or resonance.

The energy involved in the above demonstration of the goodness of the one-meson-exchange model is up to 350 MeV. On the other hand, it has also been shown that high-energy behavior[5] of the nucleon-nucleon scattering amplitude or cross section can be explained in terms of the Pomeranchuk pole, P' trajectory, and

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